The Fate of the Landau Levels under Perturbations of Constant Sign

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Frédéric Klopp, Georgi Raikov

Abstract

We show that the Landau levels cease to be eigenvalues if we perturb the 2D Schrödinger operator with constant magnetic field, by bounded electric potentials of fixed sign. We also show that, if the perturbation is not of fixed sign, then any Landau level may be an eigenvalue of the perturbed problem.

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1 Introduction. Main results

In this note we consider the Landau Hamiltonian H_0, i.e. the 2D Schrödinger operator with constant magnetic field. It is well-known that the spectrum of H_0 consists of an arithmetic progression of eigenvalues called *Landau levels* of infinite multiplicity. In Theorem 1 we show that under perturbations by fairly general electric potentials of constant sign, the Landau levels cease to be eigenvalues of the perturbed operator. Moreover, in Theorem 2 we show that for each fixed Landau level there exist non-constant-sign electric potentials such that the Landau levels is still an eigenvalue of infinite multiplicity of the perturbed operator.

Let

$$H_0 := \left(-i\frac{\partial}{\partial x} + \frac{by}{2}\right)^2 + \left(-i\frac{\partial}{\partial y} - \frac{bx}{2}\right)^2 - b$$

be the Landau Hamiltonian shifted by the value $b > 0$ of the constant magnetic field. The operator H_0 is self-adjoint in $L^2(\mathbb{R}^2)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. Note that $C_0^\infty(\mathbb{R}^2) \setminus \{0\}$ is a form core for the operator H_0. It is well-known (see [3, 6, 1]) that the spectrum $\sigma(H_0)$ of the operator H_0 consists of the so-called Landau levels $2bq$, $q \in \mathbb{N} := \{0, 1, 2 \ldots\}$, which are eigenvalues of H_0 of infinite multiplicity.

Let $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$. We will suppose that

$$c\chi(\mathbf{x}) \leq V(\mathbf{x}), \quad \mathbf{x} = (x, y) \in \mathbb{R}^2, \tag{1.1}$$

where $c > 0$ is a constant and χ is the characteristic function of a disk of radius $r > 0$ in $\mathbb{R}^2$, and

$$\|V\|_{L^\infty(\mathbb{R}^2)} < 2b. \tag{1.2}$$

Set $H_\pm := H_0 \pm V$. The main result of the note is the following

Theorem 1. *Fix $q \in \mathbb{N}$.*
(i) Assume that $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ satisfies (1.1); if $q \geq 1$, suppose in addition that (1.2) holds true. Then we have

$$\mathrm{Ker}\,(H_+ - 2bq) = \{0\}. \tag{1.3}$$

(ii) Assume that V satisfies (1.1) and (1.2). Then we have

$$\mathrm{Ker}\,(H_- - 2bq) = \{0\}. \tag{1.4}$$

The proof of Theorem 1 can be found in Section 2.

To the authors' best knowledge the fate of the Landau levels under perturbations of the described class had never been addressed in the mathematical literature. However, the asymptotic distribution of the discrete spectrum near the Landau levels of various perturbations of the Landau Hamiltonian and its generalizations has been considered by numerous authors (see [11, 4, 12, 7, 2, 10, 15, 14, 9]); in particular, it was shown in [12] that for any V which satisfies (1.1), and is relatively compact with respect to H_0, and for any Landau level there exists an infinite sequence of discrete eigenvalues of $H_\pm$ which accumulates to this Landau level. Such results are related to the problem treated here: indeed, the existence of such an infinite sequence is a necessary condition that the Landau level not be an infinitely degenerate eigenvalue of $H_\pm$.

The fact that V has a fixed sign plays a crucial role in our result, as shows the following

Theorem 2. *Fix $q \geq 0$. Then, there exists a bounded compactly supported non-constant-sign potential V such that $\|V\|_{L^\infty(\mathbb{R}^2)} < b$ and*

$$\dim\,\mathrm{Ker}\,(H_0 + V - 2bq) = \infty. \tag{1.5}$$

The proof of Theorem 2 is contained in Section 3. Its strategy is to consider radially symmetric potentials V, and, applying a decomposition into a Fourier series with respect to the angular variable, to represent the operator $H_0 + V$ as an infinite sum of ordinary differential operators involving only the radial variable. Such a representation of $H_0 + V$ is well known, and has been used in different contexts of the spectral theory of the perturbed Landau Hamiltonian (see e.g. [1, 8]). To prove Theorem 2, the basic consequence is that, for a compactly supported, radially symmetric potential V, the first derivative with respect to the coupling constant λ at $\lambda = 0$ of the eigenvalues of $H_0 + \lambda V$ close to the q-th Landau level is determined by V near the external rim of its support. Thus, writing $V = V_t$ as an infinite sum of concentric potentials depending on different coupling constant $t = (t_l)_{l\geq 1} \in \ell^\infty(\mathbb{N}^*)$, one can construct an analytic mapping from a neighborhood of 0 in $\in \ell^\infty(\mathbb{N}^*)$ to a subset of the eigenvalues of $H_0 + V_t$ near the q-th Landau level, the Jacobian of which we control for $t = 0$.

The potential exhibited in Theorem 2 can be chosen arbitrarily small. Following the same idea, one can also construct compactly supported potentials such that any of the Landau levels be of finite non trivial multiplicity or non compactly supported, bounded potentials such that (1.5) be satisfied for any $q \in \mathbb{N}$.

2 Proof of Theorem 1

Denote by Π_q, $q \in \mathbb{N}$, the orthogonal projection onto $\mathrm{Ker}(H_0 - 2bq)$. Set

$$\Pi_q^+ := \sum_{j=q}^{\infty} \Pi_j, \quad \Pi_q^- := I - \Pi_q^+, \quad q \in \mathbb{N}.$$

In order to prove Theorem 1, we need a technical result concerning some Toeplitz-type operators of the form $\Pi_q V \Pi_q$.

Lemma 3. *Let $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ satisfy (1.1). Fix $q \in \mathbb{N}$. Then*

$$\langle \Pi_q V \Pi_q u, u \rangle = 0, \quad u \in L^2(\mathbb{R}^2), \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$, implies

$$\Pi_q u = 0. \tag{2.2}$$

Proof. By (1.1) and (2.1),

$$0 \le c \langle \Pi_q \chi \Pi_q u, u \rangle \le \langle \Pi_q V \Pi_q u, u \rangle = 0, \tag{2.3}$$

i. e.

$$\langle \Pi_q \chi \Pi_q u, u \rangle = 0. \tag{2.4}$$

Denote by $T := \Pi_q \chi \Pi_q$ the operator self-adjoint in the Hilbert space $\Pi_q L^2(\mathbb{R}^2)$. The operator T is positive and compact, and its eigenvalues can be calculated explicitly (see [12, Eq. (3.32)]). This explicit calculation implies that $\mathrm{Ker}\, T = \{0\}$. Therefore, (2.2) follows from (2.4). □

Proof of Theorem 1. First, we prove (1.3) in the case $q = 0$. Assume that there exists $u \in D(H_+) = D(H_0)$ such that $H_+ u = 0$. Hence,

$$\langle H_0 u, u \rangle + \langle V u, u \rangle = 0. \tag{2.5}$$

The two terms at the l.h.s. of (2.5) are non-negative, and therefore they both should be equal to zero. Since $\langle H_0 u, u \rangle = 0$, we have

$$u = \Pi_0 u. \tag{2.6}$$

Therefore, $\langle Vu, u\rangle = \langle \Pi_0 V \Pi_0 u, u\rangle = 0$. By Lemma 3, we have $\Pi_0 u = 0$, and by (2.6) we conclude that $u = 0$.
Next, we prove (1.3) in the case $q \geq 1$. Assume that there exists $u \in D(H_0)$ such that

$$H_+ u = 2bqu. \tag{2.7}$$

Set $u_+ := \Pi_q^+ u$, $u_- := u - u_+$; evidently, $u_\pm \in D(H_0)$. Since H_0 commutes with the projections $\Pi_q^\pm$, (2.7) implies

$$H_0 u_+ - 2bqu_+ + \Pi_q^+ V \Pi_q^+ u_+ + \Pi_q^+ V \Pi_q^- u_- = 0, \tag{2.8}$$

$$H_0 u_- - 2bqu_- + \Pi_q^- V \Pi_q^- u_- + \Pi_q^- V \Pi_q^+ u_+ = 0. \tag{2.9}$$

Now note that the operator $H_0 + \Pi_q^- V \Pi_q^- - 2bq$ is boundedly invertible in $\Pi_q^- L^2(\mathbb{R}^2)$, and its inverse is a negative operator. Moreover, by (2.9) we have

$$u_- = -\left(H_0 + \Pi_q^- V \Pi_q^- - 2bq\right)^{-1} \Pi_q^- V \Pi_q^+ u_+, \tag{2.10}$$

which inserted into (2.8) implies

$$H_0 u_+ - 2bqu_+ + \Pi_q^+ V \Pi_q^+ u_+ - \Pi_q^+ V \Pi_q^- \left(H_0 + \Pi_q^- V \Pi_q^- - 2bq\right)^{-1} \Pi_q^- V \Pi_q^+ u_+ = 0,$$

and hence,

$$\begin{aligned} &\langle (H_0 - 2bq)u_+, u_+\rangle + \langle \Pi_q^+ V \Pi_q^+ u_+, u_+\rangle \\ &- \langle \Pi_q^+ V \Pi_q^- \left(H_0 + \Pi_q^- V \Pi_q^- - 2bq\right)^{-1} \Pi_q^- V \Pi_q^+ u_+, u_+\rangle = 0. \end{aligned} \tag{2.11}$$

The three terms on the l.h.s. of (2.11) are non-negative, and hence they all should be equal to zero. Since $u_+ = \Pi_q^+ u_+$, the equality $\langle (H_0 - 2bq)u_+, u_+\rangle = 0$ implies

$$u_+ = \Pi_q u_+. \tag{2.12}$$

Therefore, $\langle \Pi_q^+ V \Pi_q^+ u_+, u_+\rangle = \langle \Pi_q V \Pi_q u_+, u_+\rangle$, and $\langle \Pi_q^+ V \Pi_q^+ u_+, u_+\rangle = 0$ is equivalent to $\langle \Pi_q V \Pi_q u_+, u_+\rangle = 0$. Now by Lemma 3 we have $\Pi_q u_+ = 0$, by (2.12) we have $u_+ = 0$, and by (2.10) we have $u_- = 0$. Therefore, $u = 0$.
Finally, we sketch the proof of (1.4) which is quite similar to the one of (1.3). Let $w \in D(H_0)$, $H_- w = 2bqw$. Set $w_+ := \Pi_{q+1}^+ w$, $w_- := w - w_+$. The operator $H_0 - \Pi_{q+1}^+ V \Pi_{q+1}^+ - 2bq$ is boundedly invertible in $\Pi_{q+1}^+ L^2(\mathbb{R}^2)$, its inverse is a positive operator, and by analogy with (2.10) we get $w_+ = \left(H_0 - \Pi_{q+1}^+ V \Pi_{q+1}^+ - 2bq\right)^{-1} \Pi_{q+1}^+ V \Pi_{q+1}^- w_-$. Further, similarly to (2.11), we find that

$$\begin{aligned} &\langle (H_0 - 2bq)w_-, w_-\rangle - \langle \Pi_{q+1}^+ V \Pi_{q+1}^- w_-, w_-\rangle \\ &- \langle \Pi_{q+1}^- V \Pi_{q+1}^+ \left(H_0 - \Pi_{q+1}^+ V \Pi_{q+1}^+ - 2bq\right)^{-1} \Pi_{q+1}^+ V \Pi_{q+1}^- w_-, w_-\rangle = 0. \end{aligned}$$

The three terms on the l.h.s. are non-positive, and hence they should vanish. As in the proof of (1.3), we easily conclude that $w_- = 0$, and hence $w = 0$. $\square$

3 Proof of Theorem 2

Define the operators

$$H_0^{(m)} := -\frac{1}{\varrho}\frac{d}{d\varrho}\varrho\frac{d}{d\varrho} + \left(\frac{m}{\varrho} - b\varrho\right)^2 - b, \quad m \in \mathbb{Z},$$

self-adjoint in $L^2(\mathbb{R}_+; \varrho d\varrho)$, as the Friedrichs' extensions of the operators defined on $C_0^\infty(\mathbb{R}_+)$ with $\mathbb{R}_+ := (0, \infty)$. Then, the operator H_0 is unitarily equivalent to the orthogonal sum $\oplus_{m\in\mathbb{Z}} H_0^{(m)}$ under the passage to polar coordinates (ϱ, ϕ) in $\mathbb{R}^2$, and a subsequent decomposition into a Fourier series with respect to the angular variable ϕ. For any $m \in \mathbb{Z}$, we have

$$\sigma(H_0^{(m)}) = \bigcup_{q=m_-}^{\infty} \{2bq\}$$

where, as usual, $m_- := \max\{0, -m\}$ (see e.g. [1]). In contrast to the 2D Landau Hamiltonian H_0 however, we have $\dim \mathrm{Ker}(H_0^{(m)} - 2bq) = 1$ for all $q \geq m_-$, $m \in \mathbb{Z}$. Further, assume that $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ and V is radially symmetric i.e.

$$V(x, y) = v\left(\sqrt{x^2 + y^2}\right), \quad (x, y) \in \mathbb{R}^2.$$

Then, the operator $H_0 + V$ is unitarily equivalent to the orthogonal sum $\oplus_{m\in\mathbb{Z}}(H_0^{(m)} + v)$. Thus,

$$\dim \mathrm{Ker}(H_0 + V - 2bq) = \sum_{m\in\mathbb{Z}} \dim \mathrm{Ker}(H_0^{(m)} + v - 2bq), \quad q \in \mathbb{N}. \tag{3.1}$$

If $\|V\|_{L^\infty(\mathbb{R}^2)} = \|v\|_{L^\infty(\mathbb{R}_+)} < b$, for all $m \in \mathbb{N}$, the q-th eigenvalue of $H_0^{(m)} + v$ that we denote by $E_q(v; m)$, stays in the interval $2bq +]-b, b[$; in particular, it stays simple. So, as a consequence of regular perturbation theory, see e.g. [5, 13], the eigenvalues $(E_q(v; m))_{q\geq 0}$ are real analytic functions of the potential v. Moreover, one computes

$$\frac{\partial}{\partial t} E_q(tv; m)|_{t=0} = \int_{\mathbb{R}_+} v(\rho)\varphi_{q,m}(\varrho)^2 \varrho d\varrho \tag{3.2}$$

where

$$\varphi_{q,m}(\varrho) := \sqrt{\frac{q!}{\pi(q+m)!}\left(\frac{b}{2}\right)^{m+1}} \varrho^m L_q^{(m)}\left(b\varrho^2/2\right) e^{-b\varrho^2/4}, \quad \varrho \in \mathbb{R}_+, \quad q \in \mathbb{N},$$

are the normalized eigenfunctions of the operator $H_0^{(m)}$, $m \in \mathbb{N}$, and

$$L_q^{(m)}(s) := \sum_{l=0}^{q} \frac{(q+m)!}{(m+l)!(q-l)!}\frac{(-s)^l}{l!}, \quad s \in \mathbb{R},$$

are the generalized Laguerre polynomials.
Pick $t \in]-b/2, b/2[^{\mathbb{N}^*}$ and consider the potential

$$v_t(\rho) = -\sum_{j\in\mathbb{N}^*} t_{2j-1}\mathbf{1}_{[x_{2j-1}^-, x_{2j-1}^+]}(\rho) + \sum_{j\in\mathbb{N}^*} t_{2j}\mathbf{1}_{[x_{2j}^-, x_{2j}^+]}(\rho), \quad \rho \in \mathbb{R}_+, \tag{3.3}$$

where $x_j^- := e^{-\alpha_j/2}$, $x_j^+ := e^{-\beta_j/2}$, and

$$\alpha_{2j-1} := 2^{-N(j-1/2)^2+1},\ \beta_{2j-1} := 2^{-Nj^2+1},\ \alpha_{2j} := 2^{-N(j-1/2)^2},\ \beta_{2j} := 2^{-Nj^2}. \tag{3.4}$$

We will choose the large integer N later on.
As, for $j \geq 1$, one has

$$N(j-1)^2 < N(j-1/2)^2 - 1 < N(j-1/2)^2 < Nj^2 - 1 < Nj^2 < N(j+1/2)^2 - 1,$$

we note that, for N sufficiently large, one has:

- $\|v_t\|_{L^\infty(\mathbb{R}_+)} < b$ for $t \in]-b/2, b/2[^{\mathbb{N}^*}$;
- v_t vanishes identically if and only if the vector $(t_j)_j$ vanishes identically.

For $j \geq 1$, define $m_j = 2^{Nj^2} - 1$ and consider the mapping

$$\mathcal{E}:\ t \in]-b/2, b/2[^{\mathbb{N}^*} \mapsto (\mathcal{E}_{2j-1}(t), \mathcal{E}_{2j}(t))_{j\geq 1} = (t_{2j} + t_{2j-1}, \tilde{E}_q(v_t; m_j))_{j\geq 1} \in]-r, r[^{\mathbb{N}^*}$$

where

$$\tilde{E}_q(v_t; m_j) = \frac{2\pi q!}{C_j}\frac{m_j(m_j!)^2}{(q+m_j)!}\left(\frac{2}{b}\right)^{m_j+1}(E_q(v_t; m_j) - 2bq)$$

The constants $(C_j)_j$ are going to be chosen later on.
The mapping is real analytic and we can compute its Jacobi matrix at $t = 0$. First, bearing in mind (3.2), (3.3), and (3.4), we easily find that

$$\begin{aligned}
\partial_{t_{2j}}\mathcal{E}_{2l}(0) &= C_j^{-1}(e^{-m_l\beta_{2j}}(1+o(1)) - e^{-m_l\alpha_{2j}}(1+o(1))) \\
&= \begin{cases} 1 \text{ if } j = l, \\ O\left(e^{-2^{N|j-l|}}\right) \text{ if } l > j, \\ O\left(2^{-N|j-l|}\right) \text{ if } l < j, \end{cases} \\
\partial_{t_{2j+1}}\mathcal{E}_{2l}(0) &= -C_j^{-1}(e^{-m_l\beta_{2j+1}}(1+o(1)) - e^{-m_l\alpha_{2j+1}}(1+o(1))) \\
&= \begin{cases} -e^{-2} + O(e^{-2^{Nj}}) \text{ if } j = l, \\ O\left(e^{-2^{N|j-l|}}\right) \text{ if } l > j, \\ O\left(2^{-N|j-l|}\right) \text{ if } l < j, \end{cases}
\end{aligned}$$

when one chooses C_j properly. In this formula, $o(1)$ refers to the behavior of the function when $N \to +\infty$ uniformly in l, j. Moreover, obviously,

$$\partial_{t_{2j-1}} \mathcal{E}_{2l-1}(0) = \partial_{t_{2j}} \mathcal{E}_{2l-1}(0) = \delta_{jl}.$$

Hence, the Jacobi matrix of the mapping $\mathcal{E}(t)$ at $t = 0$ can be written as $J + E$ where J is a block diagonal matrix made of the blocks $\begin{pmatrix} 1 & 1 \\ -e^{-2} & 1 \end{pmatrix}$ and the error matrix E is a bounded operator from $l^\infty(\mathbb{N}^*)$ to itself with a norm bounded by $C2^{-N}$. So for N large enough this Jacobi matrix is invertible and, using the analytic inverse mapping theorem, we see that there exists a real analytic diffeomorphism φ on a ball of $l^\infty(\mathbb{N}^*)$ centered at 0, such that

$$\mathcal{E} \circ \varphi(u) = (u_{2j} + u_{2j-1}, u_{2j} - e^{-2} u_{2j-1})_{j \geq 1} \in]-r, r[^{\mathbb{N}^*},$$

and $\varphi(0) = 0$. To construct the potential v_t having the Landau level $2bq$ as an eigenvalue with infinite multiplicity, it suffices to take $t = \varphi(u)$ with $u_{2j} = e^{-2} u_{2j-1} \neq 0$ for infinitely many indices $j \in \mathbb{N}^*$. This completes the proof of Theorem 2.

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Frédéric Klopp
Département de mathématiques et Institut Universitaire de France
Université de Paris Nord
Avenue J.Baptiste Clément
93430 Villetaneuse, France
E-mail: klopp@math.univ-paris13.fr

G. Raikov
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Av. Vicuña Mackenna 4860
Santiago de Chile
E-mail: graikov@mat.puc.cl